Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Geometry and Physics 56 (2006) 2327–2343
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

A characteristic number of Hamiltonian bundles over S^2

Andrés Viña

Departamento de Física, Universidad de Oviedo, Avda Calvo Sotelo, 33007 Oviedo, Spain

Received 13 May 2005; received in revised form 7 October 2005; accepted 15 December 2005
Available online 20 January 2006

Abstract

Each loop ψ in the group $\mathrm{Ham}(M)$ of Hamiltonian diffeomorphisms of a symplectic manifold M determines a fibration E on S^2, whose coupling class [V. Guillemin, L. Lerman, S. Sternberg, Symplectic Fibrations and Multiplicity Diagrams, Cambridge U.P., Cambridge, 1996] is denoted by c. If VTE is the vertical tangent bundle of E, we relate the characteristic number $\int_E c_1(VTE)c^n$ to the Maslov index of the linearized flow ψ_{t*} and the Chern class $c_1(TM)$. We give the value of this characteristic number for loops of Hamiltonian symplectomorphisms of Hirzebruch surfaces.

MSC: 53D05; 57S05

Keywords: Hamiltonian diffeomorphisms; Symplectic fibrations

1. Introduction

A loop $\psi : S^1 \to \mathrm{Ham}(M, \omega)$ in the group of Hamiltonian diffeomorphisms [7] of a symplectic manifold (M^{2n}, ω) can be considered as a clutching function of a Hamiltonian fibration $E \xrightarrow{\pi} S^2$ with fibre M. The total space E supports the coupling class $c \in H^2(E, \mathbb{R})$; this is the unique class such that $c^{n+1} = 0$, and $i_p^*(c)$ is the cohomology class of the symplectic structure on the fibre $\pi^{-1}(p)$, where i_p is the inclusion of $\pi^{-1}(p)$ in E [5]. Furthermore one can consider on E the first Chern class $c_1(VTE)$ of the vertical tangent bundle of E. These canonical cohomology classes on E determine the characteristic number (see [6])

$$I_\psi = \int_E c_1(VTE)\, c^n, \tag{1.1}$$

E-mail address: vina@uniovi.es.

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2005.12.004

which depends only on the homotopy class of ψ. Since I is an $\mathbb{R}$-valued group homomorphism on $\pi_1(\mathrm{Ham}(M, \omega))$, the non-vanishing of I implies that the group $\pi_1(\mathrm{Ham}(M, \omega))$ is infinite. That is, I can be used to detect the infinitude of the corresponding homotopy group. Furthermore I calibrates the Hofer's norm ν on $\pi_1(\mathrm{Ham}(M, \omega))$ in the sense that $\nu(\psi) \geq C|I_\psi|$, for all ψ, where C is a positive constant [9].

I is a generalization of the mixed action–Maslov homomorphism introduced by Polterovich [8] for *monotone* manifolds, that is, when $[\omega] = ac_1(TM)$ and $a > 0$. The value of this mixed action–Maslov homomorphism on a loop ψ is, in many cases, easy to calculate, since it is a linear combination of the symplectic action around *any* orbit $\{\psi_t(x_0)\}_t$ and the Maslov index of the linearized flow $(\psi_t)_*$ along this orbit. By contrast, I is defined for Hamiltonian loops in general manifolds (not necessarily monotone), and its value is mostly not so easy to determine from the definition.

Our purpose in this note is to obtain an explicit expression for I_ψ, which can be used to calculate its value. More precisely, when the bundle TM admits local symplectic trivializations whose domains are fixed by the diffeomorphisms ψ_t, we deduce a formula for I_ψ in which appear a contribution related to the Maslov indices of the linearized flow ψ_{t*} in the trivializations, and a second one in which are involved transition functions of the bundle $\det(TM)$. The second contribution is related to the Chern class $c_1(TM)$ in the following sense. Using the expression of $c_1(M)$ in terms of the transition functions of TM determined by the trivializations, $\langle c_1(M)[\omega]^{n-1}, M\rangle$ can be written as a sum $\sum_j \int_{R_j} \sigma_j$, where σ_j is a $2n-1$ form (see (3.16)). It turns out that the second contribution is equal to this sum "weighted" by a multiple of the Hamiltonian f_t which generates ψ; more concretely, that contribution is $-n\sum_j \int \mathrm{d}t \int_{R_j} (f_t \circ \psi_t)\sigma_j$.

Let (M, ω, f) be an integrable system such that the points where the integrals of motion are dependent form a set P which is union of codimension 2 submanifolds of M, and such that $M \setminus P$ is invariant under ψ_t and on it there exist action-angle coordinates. Furthermore we assume that there are ψ_t invariant Darboux charts which cover P. Then the expression of I_ψ in this atlas reduces to the aforesaid second contribution; that is, $I_\psi = -n\sum_j \int_{R_j} f\sigma_j$.

The paper is organized as follows. In Section 2 we recall the construction of the coupling class c following [9]. Section 3 is concerned with the proof of the above-mentioned expression for I_ψ. First we express $\langle c_1(M)[\omega]^{n-1}, M\rangle$ as the sum $\sum_j \int_{R_j} \sigma_j$ of integrals of $2n-1$ forms, and next we use this result to prove the formula for the invariant I_ψ. In Section 4 we check and apply the formulae obtained in Section 3. Using these formulae, we calculate I_ψ, when ψ is the loop in $\mathrm{Ham}(S^2)$ generated by the 1-turn rotation of S^2 around the z-axis. The result $I_\psi = 0$ agrees with the fact that $\pi_1(\mathrm{Ham}(S^2)) = \mathbb{Z}_2$ and I is a group homomorphism on $\mathrm{Ham}(M)$. We also prove that I on $\pi_1(\mathrm{Ham}(\mathbb{T}^{2n}))$ vanishes identically. When $n = 1$ this result is consistent with the fact that $\pi_1(\mathrm{Ham}(\mathbb{T}^2)) = 0$. Finally we determine the value of I on the loops generated by action of $\mathbb{T}^2$ on a general symplectic Hirzebruch surface (see Theorem 8).

2. The coupling class

Let (M, ω) be a compact connected symplectic $2n$-manifold. Let $\psi : S^1 = \mathbb{R}/\mathbb{Z} \to \mathrm{Ham}(M, \omega)$ be a loop in the group $\mathrm{Ham}(M, \omega)$ at id. By X_t we denote the time-dependent vector field generated by ψ_t and f_t is the normalized time-dependent Hamiltonian; that is,

$$\frac{\mathrm{d}\psi_t}{\mathrm{d}t} = X_t \circ \psi_t, \quad \iota_{X_t}\omega = -\mathrm{d}f_t, \quad \int_M f_t\omega^n = 0.$$

Given ϵ, with $0 < \epsilon < \pi/2$, we set

$$D^2_+ := \{p \in S^2 | 0 \leq \theta(p) < \pi/2 + \epsilon\}$$
$$D^2_- := \{p \in S^2 | \pi/2 - \epsilon < \theta(p) \leq \pi\},$$

where $\theta \in [0, \pi]$ is the polar angle from the z-axis.

Next we construct the Hamiltonian bundle E over S^2 determined by ψ. First of all we extend ψ to a map defined on $F := D^2_+ \cap D^2_-$ by putting $\psi(\theta, \phi) = \psi_t$, with $t = \phi/2\pi$, with ϕ the spherical azimuth angle. We set

$$E = [(D^2_+ \times M) \cup (D^2_- \times M)]/\simeq, \quad \text{where}$$
$$(+, p, x) \simeq (-, p', y) \quad \text{iff} \quad \begin{cases} p = p' \in F, \\ y = \psi_t^{-1}(x), \; t = \phi(p)/2\pi. \end{cases}$$

In this way $M \hookrightarrow E \xrightarrow{\pi} S^2$ is a Hamiltonian bundle over S^2.

We assume that $D^2_\pm$ are endowed with the orientations induced by the usual one of S^2 (that is, the orientation of S^2 as a border of the unit ball). We suppose that S^1 is oriented by $\mathrm{d}t = \mathrm{d}\phi/2\pi$, that is, S^1 is oriented as ∂D_+. In E one considers the orientation induced by the one defined on $D^2_+ \times M$ by $\mathrm{d}\theta \wedge \mathrm{d}\phi \wedge \omega^n$.

Let α be a monotone smooth map $\alpha : [\pi/2 - \epsilon, \pi] \to [0, 1]$, with $\alpha(\theta) = 1$ for $\theta \in [\pi/2 - \epsilon, \pi/2 + \epsilon]$ and $\alpha(\theta) = 0$ for θ near π. Now we consider the 2-form (see [9])

$$\tau = \begin{cases} \omega, & \text{on } D^2_+ \times M \\ \omega + \mathrm{d}(\alpha(f_t \circ \psi_t)) \wedge \mathrm{d}t, & \text{on } D^2_- \times M. \end{cases} \tag{2.1}$$

As α vanishes near π, τ is well defined on $D^2_- \times M$; moreover on $F \times M \subset D^2_- \times M$, τ reduces to $\omega + \mathrm{d}(f_t \circ \psi_t) \wedge \mathrm{d}t$. If we denote by h the map

$$h : F \times M \subset D^2_- \times M \to F \times M \subset D^2_+ \times M$$

given by $h(p, x) = (p, \psi_t(x))$, with $t = \phi(p)/2\pi$, then taking into account that $h_*(\frac{\partial}{\partial t}) = \frac{\partial}{\partial t} + X_t \circ \psi_t$, it follows from $\iota_{X_t}\omega = -\mathrm{d}f_t$ that $h^*\omega = \omega + \mathrm{d}(f_t \circ \psi_t) \wedge \mathrm{d}t$. So one has the following proposition.

Proposition 1. *τ defines a closed 2-form on E.*

Moreover the cohomology class $[\tau] \in H^2(E, \mathbb{R})$ restricted to each fibre coincides with $[\omega]$. On the other hand

$$\int_E \tau^{n+1} = (n+1) \int_{D^2_- \times M} (f_t \circ \psi_t)\alpha'(\theta)\mathrm{d}\theta \wedge \mathrm{d}t \wedge \omega^n.$$

From the normalization condition for f_t it follows that $\int_E \tau^{n+1} = 0$. Hence $[\tau]$ is the coupling class c of the fibration E [5,7].

3. The characteristic number I_ψ

Defining $TM = \{v_x \in T_x M | x \in M\}$, we put

$$VTE = [(D^2_+ \times TM) \cup (D^2_- \times TM)]/\simeq,$$

with

$$(+, p, v_x) \simeq (-, p', v'_{x'}) \quad \text{iff } p = p',\ x' = \psi_t^{-1}(x),\ v'_{x'} = (\psi_t^{-1})_*(v_x)$$

where $t = \phi(p)/2\pi$. So *VTE* is a vector bundle over E; by construction it is the vertical tangent bundle of E.

Let $(U; X_1, \ldots, X_{2n})$ be a symplectic trivialization of *TM* on $U \subset M$, and $(V; Y_1, \ldots, Y_{2n})$ be a symplectic trivialization on $V \subset M$. We put

$$U_\pm := \{[\pm, p, x] | p \in D^2_\pm, x \in U\} \tag{3.1}$$

and similarly for $V_\pm$. Defining $x_t := \psi_t^{-1}(x)$ one has

$$U_+ \cap U_- = \{[+, p, x] | p \in F,\ x \in U,\ x_t \in U\}$$
$$V_+ \cap V_- = \{[+, p, x] | p \in F,\ x \in V,\ x_t \in V\}$$
$$V_- \cap U_- = \{[-, p, x] | p \in D^2_-,\ x \in V \cap U\}$$
$$U_+ \cap V_+ = \{[+, p, x] | p \in D^2_+,\ x \in V \cap U\}.$$

The corresponding transition functions of *VTE* are

$$g_{U_-U_+}([+, p, x]) = A(t, x) \in Sp(2n, \mathbb{R}), \quad \text{with } \psi_{t*}^{-1}(X_i(x)) = \sum_k A^k{}_i(t, x)\, X_k(x_t)$$
$$g_{V_-V_+}([+, p, x]) = B(t, x) \in Sp(2n, \mathbb{R}), \quad \text{with } \psi_{t*}^{-1}(Y_i(x)) = \sum_k B^k{}_i(t, x)\, Y_k(x_t)$$
$$g_{U_-V_-}([-, p, x]) = R(x) = g_{U_+V_+}([+, p, x]), \quad \text{with } Y_i(x) = \sum_k R^k{}_i(x)\, X_k(x).$$

We denote by ρ the usual map $\rho : Sp(2n, \mathbb{R}) \to U(1)$ which restricts to the determinant map on $U(n)$ [10], then $l_{ab} := \rho \circ g_{ab}$ is a transition function for det(*VTE*). We also use the following notation, the matrices in $Sp(2n, \mathbb{R})$ are denoted with capital letters and its images by ρ will be denoted by the corresponding small letters; that is,

$$a(t, x) := \rho(A(t, x)), \quad b(t, x) := \rho(B(t, x)), \quad r_{UV}(x) := \rho(R(x)). \tag{3.2}$$

If $\psi_t(U) \subset U$ for all t, given $x \in U$, the winding number of the map $t \in S^1 \mapsto a^{-1}(t, x) \in U(1)$ is the integer

$$\frac{i}{2\pi} \int_0^1 a^{-1}(t, x) \frac{\partial a}{\partial t}(t, x)\, \mathrm{d}t. \tag{3.3}$$

This integer is independent of the point $x \in U$, it will be denoted as J_U. The number J_U is the Maslov index in U of the linearized flow ψ_{t*}. Analogously, if $\psi_t(V) \subset V$ for all t we have the integer

$$J_V = \frac{i}{2\pi} \int_0^1 b^{-1}(t, x) \frac{\partial b}{\partial t}(t, x)\, \mathrm{d}t, \tag{3.4}$$

x being any point of V; this is the Maslov index in V of ψ_{t*}.

As a step towards computing I_ψ we shall prove the following lemma, in which the value $\langle c_1(M)[\omega]^{n-1}, [M]\rangle$ is expressed in terms of transition functions of det(*TM*).

Lemma 2. *Let* $\{B_1, \ldots, B_m\}$ *be a set of trivializations of* TM*, such that its domains cover* M*. Then*

$$\langle c_1(TM)[\omega]^{n-1},\ [M]\rangle = \frac{-i}{2\pi}\sum_{i<k}\int_{A_{ik}} \mathrm{d}(\log s_{ik})\wedge\omega^{n-1}, \tag{3.5}$$

s_{ik} *being the corresponding transition function of* $\det(TM)$ *and*

$$A_{ik} = (\partial B_i \setminus \cup_{r<k} B_r)\cap B_k. \tag{3.6}$$

Proof. $c_1(M)$ is represented on B_a by the 2-form

$$\frac{-i}{2\pi}\sum_c \mathrm{d}(\varphi_c \mathrm{d}\log s_{ac}),$$

where $\{\varphi_c\}$ is a partition of unity subordinate to the covering $\{B_1, \ldots, B_m\}$.

If $m = 2$

$$\langle c_1(M)[\omega]^{n-1},\ [M]\rangle = \frac{-i}{2\pi}\int_{B_1}\mathrm{d}(\varphi_2 \mathrm{d}\log s_{12})\wedge\omega^{n-1} + \frac{-i}{2\pi}\int_{B_2\setminus B_1}\mathrm{d}(\varphi_1 \mathrm{d}\log s_{21})\wedge\omega^{n-1}.$$

By Stokes's theorem

$$\langle c_1(M)[\omega]^{n-1},\ [M]\rangle = \int_{\partial B_1}\varphi_2 L_{12} + \int_{\partial(B_2\setminus B_1)}\varphi_1 L_{21}, \tag{3.7}$$

where

$$L_{jk} := (-i/2\pi)\mathrm{d}\log s_{jk}\wedge\omega^{n-1}.$$

Since $\partial(B_2 \setminus B_1)\cap B_1 = \emptyset$, φ_1 vanishes on $\partial(B_2\setminus B_1)$ and the last integral in (3.7) is zero.

As φ_2 is 1 on ∂B_1, we have

$$\langle c_1(M)[\omega]^{n-1},\ [M]\rangle = \int_{\partial B_1} L_{12}.$$

In this case $\partial B_1 \subset B_2$, so $\partial B_1 = A_{12}$, and the lemma is proved when $m = 2$.

If $m = 3$

$$\langle c_1(M)[\omega]^{n-1},\ [M]\rangle = \int_{\partial B_1}(\varphi_2 L_{12} + \varphi_3 L_{13}) + \int_{\partial(B_2\setminus B_1)}(\varphi_1 L_{21} + \varphi_3 L_{23}) \tag{3.8}$$

$$+ \int_{\partial(B_3\setminus(B_1\cup B_2))}(\varphi_1 L_{31} + \varphi_2 L_{32}). \tag{3.9}$$

As $\partial(B_3 \setminus (B_1 \cup B_2))$ and the interior of $B_1 \cup B_2$ are disjoint sets, φ_1 and φ_2 vanish on $\partial(B_3 \setminus (B_1 \cup B_2))$, and the integral in (3.9) is zero. Analogously $\partial(B_2 \setminus B_1)$ and the support of φ_1 are disjoint so

$$\langle c_1(M)[\omega]^{n-1},\ [M]\rangle = \int_{\partial B_1}(\varphi_2 L_{12} + \varphi_3 L_{13}) + \int_{\partial(B_2\setminus B_1)}\varphi_3 L_{23}. \tag{3.10}$$

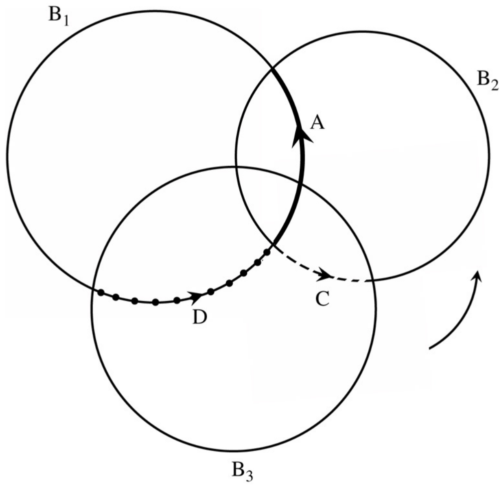

Fig. 1. $A = \partial B_1 \cap B_2$, $D = (\partial B_1 \setminus B_2) \cap B_3$ and $C = (\partial B_2 \setminus B_1) \cap B_3$.

On the other hand $\partial B_1 = A + D$, with $A := \partial B_1 \setminus B_2$ (oriented as ∂B_1) and $D := (\partial B_1 \setminus B_2) \cap B_3$ (see Fig. 1). Moreover $\partial(B_2 \setminus B_1) = -A + C$ with $C := (\partial B_2 \setminus B_1) \cap B_3$ (oriented as ∂B_2).

Since $C \cap (B_1 \cup B_2) = \emptyset$, then $\varphi_3|_C = 1$; thus

$$\langle c_1(M)[\omega]^{n-1}, [M]\rangle = \int_{A+D} (\varphi_2 L_{12} + \varphi_3 L_{13}) + \int_{-A} \varphi_3 L_{23} + \int_{A_{23}} L_{23}. \tag{3.11}$$

The last integral in (3.11) is just the term in (3.5) with $i = 2, k = 3$.

Since $\varphi_j|_D = 0$, for $j = 1, 2$, then $\varphi_3|_D = 1$. As A and the support of φ_1 are disjoint sets, then $(\varphi_2 + \varphi_3)|_A = 1$. It follows from these facts together with the cocycle condition $L_{13} + L_{32} = L_{12}$ that

$$\langle c_1(M)[\omega]^{n-1}, [M]\rangle = \int_A L_{12} + \int_D L_{13} + \int_{A_{23}} L_{23}. \tag{3.12}$$

On the other hand $A_{12} = (\partial B_1 \setminus B_1) \cap B_2 = A$. Similarly $A_{13} = D$. Therefore (3.12) is the formula given in the statement of lemma when $m = 3$.

The preceding arguments can be generalized to any m

$$\langle c_1(TM)[\omega]^{n-1}, [M]\rangle = \int_{\partial B_1} \sum_{j\neq 1} \varphi_j L_{1j} + \cdots + \int_{\partial(B_{m-1}\setminus \cup_{r<m-1} B_r)} \sum_{j\neq m-1} \varphi_j L_{m-1,j} \tag{3.13}$$

$$+ \int_{\partial(B_m \setminus \cup_{r<m} B_r)} \sum_{j\neq m} \varphi_j L_{m-1,j}. \tag{3.14}$$

For any $j = 1, \ldots, m-1$ the support of φ_j and $\partial(B_m \setminus \cup_{r<m} B_r)$ are disjoint sets. Thus the integral (3.14) is zero (as in the cases $m = 2, 3$). We decompose

$$\partial(B_{m-1} \setminus \cup_{r<m-1} B_r) = E + G,$$

with

$$E := (\partial B_{m-1} \setminus \cup_{r<m-1} B_r) \cap B_m.$$

Then $\varphi_j|_E = 0$ for all $j \neq m$ and $\varphi_m|_E = 1$; thus

$$\int_{\partial(B_{m-1}\setminus\cup_{r<m-1} B_r)} \sum_{j\neq m-1} \varphi_j L_{m-1,j} = \int_G + \int_{A_{m-1,m}} L_{m-1,m}. \tag{3.15}$$

The last integral in (3.15) is the term in (3.5) which corresponds to $i = m-1, k = m$. A calculation analogous to, but more tedious than, the one for the case $m = 3$ allows us to identify in (3.13) the remainder terms of (3.5). □

Lemma 2 gives a way of expressing $\langle c_1(TM)[\omega]^{n-1},\ [M]\rangle$ as a sum of integrals of $2n-1$ differential forms on $2n-1$ chains. The right-hand side of (3.5) can be written schematically as

$$\sum_j \int_{R_j} \sigma_j. \tag{3.16}$$

In the next theorem we use this expression to give an explicit formula for I_ψ in terms of transition functions of $\det(TM)$ and Maslov indices of ψ_{t*}.

Theorem 3. *If $\{B_1, \ldots, B_m\}$ is a set of symplectic trivializations for TM which covers M, and such that $\psi_t(B_j) = B_j$, for all t and all j, then*

$$I_\psi = \sum_{i=1}^{m} J_i \int_{B_i\setminus\cup_{j<i} B_j} \omega^n + \sum_{i<k} N_{ik}, \tag{3.17}$$

where

$$N_{ik} = n\frac{i}{2\pi}\int_0^1 \mathrm{d}t \int_{A_{ik}} (f_t \circ \psi_t)(\mathrm{d}\log r_{ik}) \wedge \omega^{n-1},$$

$A_{ik} = (\partial B_i \setminus \cup_{r<k} B_r) \cap B_k$, *$J_i$ is the Maslov index of $(\psi_t)_*$ in the trivialization B_i and r_{ik} the corresponding transition function of* $\det(TM)$.

Proof. Using the notation (3.1) we put

$$O_{\mathbf{2a-1}} := (B_a)_-, \quad O_{\mathbf{2a}} := (B_a)_+. \tag{3.18}$$

Then $\{O_{\mathbf{c}} | c = 1, \ldots, 2m\}$ is a covering for E. We shall denote by $l_{\mathbf{bc}}$ the respective transition functions for $\det(VTE)$. If we set $U := B_1,\ V := B_2$, one has by (3.2)

$$l_{\mathbf{12}} = a(t,x), \quad l_{\mathbf{13}} = r_{UV}(x), \quad l_{\mathbf{34}} = b(t,x).$$

We can determine $I_\psi = \langle c_1(VTE)\, c^n,\ [E]\rangle$ applying the result given in Lemma 2 to the set $\{O_{\mathbf{c}}\}$ of trivializations of VTE. That is,

$$I_\psi = \sum_{\mathbf{a}<\mathbf{b}} \mathcal{T}_{\mathbf{ab}}, \quad \text{where } \mathcal{T}_{\mathbf{ab}} = \frac{-i}{2\pi}\int_{A_{\mathbf{ab}}} \mathrm{d}\log l_{\mathbf{ab}} \wedge \tau^n. \tag{3.19}$$

It follows from (3.18) and (2.1) that τ is equal to ω on $A_{\mathbf{ab}}$ unless **a** and **b** are both odd; in this case $\tau = \omega + \mathrm{d}(\alpha(f_t \circ \psi_t)) \wedge \mathrm{d}t$.

We will calculate the summand $\mathcal{T}_{\mathbf{12}}$ in (3.19). The set $A_{\mathbf{12}} = \partial O_{\mathbf{1}} \cap O_{\mathbf{2}} = \partial U_- \cap U_+$, and

$$\partial U_- = \{[+, p, x] | p \in \partial D^2_-,\ x \in U\} \cup \{[-, p, x] | p \in D^2_-,\ x \in \partial U\}.$$

So

$$A_{\mathbf{12}} = \{[+, p, x] | p \in \partial D^2_-,\ x \in U\}.$$

Taking into account (3.3) and (3.2) together with the fact that orientations of S^1 and ∂D^2_- are opposite, we deduce

$$\mathcal{T}_{\mathbf{12}} = \frac{-i}{2\pi} \int_U \left(\int_{-S^1} a^{-1}(t, x) \frac{\partial a(t, x)}{\partial t} \, \mathrm{d}t \right) \omega^n = J_U \int_U \omega^n.$$

Next we consider the term $\mathcal{T}_{\mathbf{34}}$. The integration domain is

$$A_{\mathbf{34}} = (\partial V_- \setminus (U_- \cup U_+)) \cap V_+ = \{[+, p, x] | p \in \partial D^2_-,\ x \in V \setminus U\}.$$

Hence

$$\mathcal{T}_{\mathbf{34}} = J_V \int_{V \setminus U} \omega^n.$$

In general,

$$A_{\mathbf{2j-1,2j}} = \left(\partial B_{j-} \setminus \cup_{r<j}(B_{r+} \cup B_{r-})\right) \cap B_{j+} = \{[+, p, x] | p \in \partial D^2_-,\ x \in B_j \setminus \cup_{r<j} B_r\}.$$

Hence the term in (3.19) with $\mathbf{a} = \mathbf{2j-1}$, $\mathbf{b} = \mathbf{2j}$ gives a contribution to I_ψ equal to

$$J_{B_j} \int_{B_j \setminus \cup_{r<j} B_r} \omega^n. \tag{3.20}$$

Now we analyze $\mathcal{T}_{\mathbf{13}}$.

$$A_{\mathbf{13}} = \{[-, p, x] | p \in D^2_-,\ x \in \partial U \cap V\},$$

Here D^2_- is oriented by the form $\mathrm{d}\theta \wedge \mathrm{d}t$, and $\partial U \cap V$ is oriented with the orientation of ∂U. Hence

$$\begin{aligned}
\mathcal{T}_{\mathbf{13}} &= \frac{-i}{2\pi} \int_{A_{\mathbf{13}}} \mathrm{d} \log r_{UV}(x) \wedge (\omega + \mathrm{d}(\alpha(f_t \circ \psi_t)) \wedge \mathrm{d}t)^n \\
&= \frac{-ni}{2\pi} \int_{A_{\mathbf{13}}} \mathrm{d} \log r_{UV}(f_t \circ \psi_t) \alpha'(\theta) \mathrm{d}\theta \wedge \mathrm{d}t \wedge \omega^{n-1} \\
&= \frac{+ni}{2\pi} \int_0^1 \mathrm{d}t \int_{\partial U \cap V} (f_t \circ \psi_t) \mathrm{d} \log r_{UV} \wedge \omega^{n-1}.
\end{aligned} \tag{3.21}$$

In general, if $j < k$, then

$$\mathcal{T}_{\mathbf{2j-1,2k-1}} = \frac{ni}{2\pi} \int_0^1 \mathrm{d}t \int_{A_{jk}} (f_t \circ \psi_t) \, \mathrm{d} \log r_{jk} \wedge \omega^{n-1}, \tag{3.22}$$

where A_{jk} is the set defined in Lemma 2.

On the other hand

$$A_{\mathbf{14}} = (\partial U_- \setminus (U_+ \cup V_-)) \cap V_+ = \{[-, p, x] | p \in D^2_-,\ x \in \partial U \setminus V\} \cap V_+ = \emptyset.$$

Thus $\mathcal{T}_{\mathbf{14}} = 0$. In general, for $j < k$ the integration domain $A_{\mathbf{2j-1,2k}}$ is of the form

$$(\partial B_{j-} \setminus \cup \cdot) \cap B_{k+}.$$

In the union $\cup\cdot$ there appear the sets B_{k-} and B_{j+}, hence

$$A_{\mathbf{2j-1,2k}} \subset (\partial B_{j-} \setminus (B_{j+} \cup B_{k-})) \cap B_{k+},$$

and this set is empty for the same reason that $A_{\mathbf{14}} = \emptyset$. Therefore $\mathcal{T}_{\mathbf{2j-1,2k}} = 0$, for any $j < k$.

The set $A_{\mathbf{23}}$ is

$$A_{\mathbf{23}} = (\partial U_+ \setminus U_-) \cap V_- = \{[+, p, x] | p \in F,\ x \in \partial U \setminus V\}.$$

As $\mathrm{d}\log l_{\mathbf{23}} \wedge \omega^n$ does not contain $\mathrm{d}\theta$, the term $\mathcal{T}_{\mathbf{23}}$ vanishes. In general, if $j < k$

$$\begin{aligned} A_{\mathbf{2j,2k-1}} &= (\partial B_{j+} \setminus \cup \cdot) \cap B_{k-} \subset (\partial B_{j+} \setminus B_{j-}) \cap B_{k-} \\ &= \{[+, p, x] | p \in F,\ x \in \partial B_j \setminus B_k\}. \end{aligned}$$

Then $\mathcal{T}_{\mathbf{2j,2k-1}}$ vanishes by the same reason that $\mathcal{T}_{\mathbf{23}} = 0$.

Analogous arguments as the ones explained in the preceding paragraph show that $\mathcal{T}_{\mathbf{2j,2k}} = 0$, for any $j < k$.

So, apart from the terms $\mathcal{T}_{\mathbf{ab}}$ considered in (3.20) and in (3.22), the remainder summands in (3.19) are zero. The theorem follows from (3.20) and (3.22). □

From the definition of the product in $\pi_1(\mathrm{Ham}(M, \omega))$ by juxtaposition of paths and under the hypotheses of Theorem 3 it is obvious that

$$I : \pi_1(\mathrm{Ham}(M, \omega)) \to \mathbb{R}$$

is a group homomorphism. This fact has been proved in [6] for the general case.

Corollary 4. *If U and V are symplectic trivializations of TM, with $\psi_t(U) = U$, $\psi_t(V) = V$, for all t and $U \cup V = M$ and $\int_{S^1} (f_t \circ \psi_t)\mathrm{d}t$ is a constant k on $\partial U \cap V$, then*

$$I_\psi = J_U \int_U \omega^n + J_V \int_{V \setminus U} \omega^n - nk\langle c_1(TM)[\omega]^{n-1},\ M\rangle.$$

Corollary 5. *If TM is trivial on $U := M \setminus \{q\}$, where q is a point of M fixed by ψ_t for all t, then*

$$I_\psi = J_U \int_M \omega^n - n \left(\int_{S^1} f_t(q)\mathrm{d}t \right) \langle c_1(TM)[\omega]^{n-1},\ M\rangle.$$

Now we analyze the expression for I_ψ given in Theorem 3 in the case of *integrable systems*. Let f be the normalized Hamiltonian which generates the loop ψ. We assume that (M, ω, f) is completely integrable, with $f_1 = f, f_2, \ldots, f_n$ integrals of motion. We suppose that $\mathrm{d}f_1, \ldots, \mathrm{d}f_n$ are independent at the points of $M \setminus P =: V$, where P is a finite union of $2n-2$ dimensional submanifolds of M. We suppose that on V are defined action-angle coordinates [1]. We put

$$Q := \{x \in P | \dim \mathrm{Span}\ (\mathrm{d}f_1(x), \ldots, \mathrm{d}f_n(x)) = n - 1\}.$$

By $Q_1, \ldots, Q_k$ we denote the connected components of Q, and let V_j be a tubular neighborhood of Q_j in M, invariant under ψ_t for all t. We assume that on V_j is defined a symplectic trivialization of TM. Then, for each j one can choose a family of tubular neighborhoods $\{V_{jb} \subset V_j\}_{b=1,2\dots}$, such that

$$\lim_{b\to\infty} \int_{V_{jb}} \omega^n = 0.$$

Lemma 2 applied to the covering $\{V, V_{jb}\}_{j=1,\dots,k}$ of $V \cup Q$ gives

$$\langle c_1(M)[\omega]^{n-1}, [M]\rangle = \frac{-i}{2\pi} \sum_{j=1}^{k} \int_{\partial V \cap V_{jb}} \mathrm{d}\log r_{VV_{jb}} + \epsilon(b),$$

where $\epsilon(b)$ goes to 0 as $b \to \infty$.

Hence

$$\langle c_1(M)[\omega]^{n-1}, [M]\rangle = \sum_{j=1}^{k} z_j, \tag{3.23}$$

with

$$z_j := \frac{-i}{2\pi} \sum \int_{\partial V \cap V_{jb}} \mathrm{d}\log r_{VV_{jb}} \wedge \omega^{n-1}. \tag{3.24}$$

Proposition 6. *Let $(M, \omega, f, f_2, \dots, f_n)$ be an integrable system in which the preceding hypotheses hold, then*

$$I_\psi = \sum_{j=1}^{k} z'_j,$$

where z'_j is obtained from the corresponding z_j by inserting the factor $-nf$ in the integrand of (3.24).

Proof. The Maslov index $J_V = 0$ because of the particular form of the flow equations in action-angle coordinates. On the other hand

$$\int_{V_{jb}\setminus(V\cup\cdots)} \omega^n = 0.$$

Thus the proposition follows from Theorem 3, together with (3.23) and (3.24). □

Arguments similar to the ones involved in this proposition are used in Section 4 for studying the invariant I in Hirzebruch surfaces.

4. Examples

The invariant I when the manifold is the 2-sphere.

Let ψ_t be the rotation in $\mathbb{R}^3$ around $\vec{e}_3$ of angle $2\pi t$ with $t \in [0, 1]$. Then ψ_t determines a Hamiltonian symplectomorphism of $(S^2, \omega_{\text{area}})$. In fact, the isotopy $\{\psi_t\}$ is generated by the vector field $\frac{\partial}{\partial\phi}$, and the function f on S^2 defined by $f(\theta, \phi) = -2\pi \cos\theta = -2\pi z$ is the corresponding normalized Hamiltonian.

TS^2 can be trivialized on $U = D^2_+$, and on $V = D^2_-$. Moreover $\partial U \cap V$ is the parallel $\theta = \pi/2 + \epsilon$. On $\partial U \cap V$ the function $f \circ \psi_t$ takes the value $2\pi \sin\epsilon$.

$$\int_U \omega = 2\pi(1-k'), \quad \int_{V\setminus U} \omega = 2\pi(1+k'),$$

with $k' := \cos(\pi/2+\epsilon)$.

Furthermore the north pole n and the south pole s are fixed points of the isotopy ψ_t. The rotation ψ_t transforms the basis $\vec{e}_1, \vec{e}_2$ of $T_n S^2$ in

$$(\cos 2\pi t\, \vec{e}_1 + \sin 2\pi t\, \vec{e}_2, \ -\sin 2\pi t\, \vec{e}_1 + \cos 2\pi t\, \vec{e}_2).$$

So J_U is the winding number of the map

$$t \in [0,\ 1] \to e^{2\pi t i} \in U(1);$$

That is, $J_U = +1$.

Similarly, by considering the oriented basis $\vec{e}_2, \vec{e}_1$ of $T_s S^2$ it turns out that the Maslov index J_V of ψ_t is -1.

By Corollary 4

$$I_\psi = 2\pi(1-k') - 2\pi(1+k') - (-2\pi k')\langle c_1(TS^2),\ S^2\rangle = 0.$$

Corollary 5 can also be applied to determine I_ψ. One takes $U := S^2 \setminus \{s\}$. As $f(s) = -2\pi(-1)$, we obtain again

$$I_\psi = +4\pi - 2\pi\langle c_1(TS^2),\ S^2\rangle = 0.$$

Using formula (3.24) we can determine I_ψ again. Now V is $S^2 \setminus \{n, s\}$, U_1 is a small polar cap at n and U_2 the symmetric one at s. By the symmetry

$$\int_{\partial U_1 \cap V} \mathrm{d}\log r_{U_1 V} = \int_{\partial U_2 \cap V} \mathrm{d}\log r_{U_2 V}$$

so $z_1 = z_2$. As $f(n) = -f(s)$, we have $I_\psi = 0$.

This result was expected, because $\pi_1(\mathrm{Ham}(S^2))$ is isomorphic to $\mathbb{Z}_2$ (see [9]) and I is a group homomorphism.

The invariant I for Hamiltonian loops in $\mathbb{T}^{2n}$.

We identify the torus $\mathbb{T}^{2n}$ with $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$, and we suppose that $\mathbb{T}^{2n}$ is equipped with the standard symplectic form ω_0. If ψ_t is a Hamiltonian isotopy of $\mathbb{T}^{2n}$, it can be written in the form

$$\psi_t(x^1, \ldots, x^{2n}) = (x^1 + \alpha^1(t, x^i), \ldots, x^{2n} + \alpha^{2n}(t, x^i)),$$

where the function α^j, for $j = 1, \ldots, 2n$, is periodic of period 1 in each variable: $t, x^1, \ldots, x^{2n}$. The vector fields $\{\frac{\partial}{\partial x^i}\}$ give a symplectic trivialization of the tangent bundle. In this case the right-hand side of (3.17) has only one term. The matrix of $(\psi_t)_*$ with respect to $\{\frac{\partial}{\partial x^i}\}$ is

$$\left(\delta_i^j + \frac{\partial \alpha^j}{\partial x^i}\right) \in Sp(2n,\ \mathbb{R}). \tag{4.1}$$

First, let us assume that each α^j is a separate variables function; that is, $\alpha^j(t, x^i) = f^j(t)u^j(x^i)$. Since α^1 takes the same value at symmetric points on opposite faces of the cube I^{2n}, there is a point $p_1 \in I^{2n}$ such that

$$\frac{\partial u^1}{\partial x^j}(p_1) = 0,$$

for all j. Hence the first row of the matrix (4.1) at the point p_1 is $(1, 0, \dots, 0)$; that is, the matrix of $(\psi_t)_*(p_1)$ is independent of f^1 and thus the Maslov index of $\{(\psi_t)_*(p_1)\}_t$ does not depend on f^1. From (3.17) it follows that I_ψ is independent of f^1. The independence of I_ψ with respect to f^i is proved in a similar way. Thus in order to determine I_ψ we can assume that $f^i = 0$ for all i, but in this case $I_\psi = 0$ obviously.

If α^j is sum of two separate variables functions

$$\alpha^j(t, x^i) = f^j(t)u^j(x^i) + g^j(t)v^j(x^i),$$

we take a point $q_1 \in I^{2n}$, such that $\frac{\partial v^1}{\partial x^j}(q_1) = 0$, for all j. Then I_ψ is independent of g^1. The above reasoning gives $I_\psi = 0$ in this case as well.

By the Fourier theory, the original C^∞ periodic function α^j can be approximated (in the uniform C^k-norm) by a sum of separated functions of the form $\sum f_a(t)u_a(x^i)$, where f_a and u_a are 1-periodic. As I_ψ depends only on the homotopy class of ψ, we conclude that $I_\psi = 0$ for a general Hamiltonian loop.

Proposition 7. *The invariant I is identically zero on $\pi_1(\mathrm{Ham}(\mathbb{T}^{2n}, \omega_0))$.*

This result when $n = 1$ is consistent with the fact that $\pi_1(\mathrm{Ham}(\mathbb{T}^2)) = 0$ (see [9]).

Application to Hirzebruch surfaces.

Given 3 numbers k, τ, μ, with $k \in \mathbb{Z}_{>0}$, $\tau, \mu \in \mathbb{R}_{>0}$ and $k\mu < \tau$, the triple (k, τ, μ) determine a Hirzebruch surface $M_{k,\tau,\mu}$ [3]. This manifold is the quotient

$$\{z \in \mathbb{C}^4 : k|z_1|^2 + |z_2|^2 + |z_4|^2 = \tau/\pi,\ |z_1|^2 + |z_3|^2 = \mu/\pi\}/\mathbb{T}^2,$$

where the $\mathbb{T}^2$-action is given by

$$(a, b) \cdot (z_1, z_2, z_3, z_4) = (a^k b z_1, a z_2, b z_3, a z_4),$$

for $(a, b) \in \mathbb{T}^2$. The map

$$[z_1, z_2, z_3, z_4] \mapsto ([z_2 : z_4], [z_2^k z_3 : z_4^k z_3 : z_1])$$

allows us to represent $M_{k,\tau,\mu}$ as a submanifold of $\mathbb{C}P^1 \times \mathbb{C}P^2$. On the other hand the usual symplectic structures on $\mathbb{C}P^1$ and $\mathbb{C}P^2$ induce a symplectic form ω on $M_{k,\tau,\mu}$, and the following $\mathbb{T}^2$-action on $\mathbb{C}P^1 \times \mathbb{C}P^2$

$$(a, b)([u_0 : u_1], [x_0 : x_1 : x_2]) = ([a u_0 : u_1], [a^k x_0 : x_1 : b x_2])$$

gives rise to a toric structure on $M_{k,\tau,\mu}$. In terms of the Delzant construction $(M_{k,\tau,\mu}, \omega)$ is associated with the trapezoid in $(\mathbb{R}^2)^*$ whose non-oblique edges are $\tau, \mu, \lambda := \tau - k\mu$, [4] (see Fig. 2). Moreover λ is the value that the symplectic form ω takes on the exceptional divisor, $\{[z] \in M | z_3 = 0\}$, of $M := M_{k,\tau,\mu}$. And ω takes the value μ on the class of the fibre in the fibration $M \to \mathbb{C}P^1$.

Since M is a toric manifold, the $\mathbb{T}^2$-action defines symplectomorphisms of M. More precisely, let ψ_t be the diffeomorphism of M defined by

$$\psi_t[z_1, z_2, z_3, z_4] = [z_1 e^{2\pi i t}, z_2, z_3, z_4]. \tag{4.2}$$

$\psi = \{\psi_t : t \in [0, 1]\}$ is a loop of Hamiltonian symplectomorphisms of (M, ω). Similarly we have

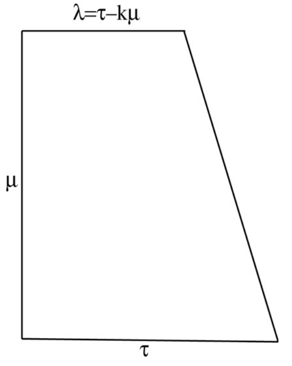

Fig. 2. Delzant polytope associated with M.

$$\tilde{\psi}_t[z_1, z_2, z_3, z_4] = [z_1, z_2 e^{2\pi i t}, z_3, z_4], \tag{4.3}$$

and the corresponding loop $\tilde{\psi}$ in $\mathrm{Ham}(M, \omega)$.

Using Theorem 3 we shall calculate the values of I_ψ and $I_{\tilde{\psi}}$ in terms of λ, τ and k. The result is stated in Theorem 8 below. The most laborious point in the proof of the following theorem is obtaining Darboux charts for M which give rise to simple transition functions for $\det(TM)$.

Theorem 8. *Let ψ and $\tilde{\psi}$ be the loops of symplectomorphisms of the Hirzebruch surface $(M_{k,\tau,\mu}, \omega)$, defined by (4.2) and (4.3) respectively, then*

$$I_\psi = \frac{2k\mu^2}{3}\left(1 - \frac{\mu}{2\lambda + k\mu}\right), \quad \text{and} \quad I_{\tilde{\psi}} = \frac{-k^2\mu^2}{3}\left(1 - \frac{\mu}{2\lambda + k\mu}\right).$$

λ being $\tau - k\mu$.

Proof. We will define a Darboux atlas on M. First we consider the following covering for M

$$U_1 = \{[z] \in M : z_3 \neq 0 \neq z_4\}, \quad U_2 = \{[z] \in M : z_1 \neq 0 \neq z_4\}$$
$$U_3 = \{[z] \in M : z_1 \neq 0 \neq z_2\}, \quad U_4 = \{[z] \in M : z_2 \neq 0 \neq z_3\}.$$

We set $z_j = \rho_j e^{i\theta_j}$, with $\rho_j = |z_j|$, and on U_1 introduce the coordinates (x_1, y_1, a_1, b_1) through the formulae

$$x_1 + iy_1 = \rho_1 e^{i\varphi_1}, \quad a_1 + ib_1 = \rho_2 e^{i\varphi_2}, \quad \varphi_1 = \theta_1 - \theta_3 - k\theta_4, \quad \varphi_2 = \theta_2 - \theta_4.$$

Then ω on U_1 can be written as $\omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}a_1 \wedge \mathrm{d}b_1$.

On U_2 we consider the Darboux coordinates (x_2, y_2, a_2, b_2), with

$$x_2 + iy_2 = \rho_3 e^{i\xi_3}, \quad a_2 + ib_2 = \rho_2 e^{i\xi_2}, \quad \xi_2 = \theta_2 - \theta_4, \quad \xi_3 = \theta_3 - \theta_1 + k\theta_4.$$

On U_3 we put

$$x_3 + iy_3 = \rho_3 e^{i\chi_3}, \quad a_3 + ib_3 = \rho_4 e^{i\chi_4}, \quad \chi_3 = \theta_3 - \theta_1 + k\theta_2, \quad \chi_4 = \theta_4 - \theta_2,$$

and $\omega = \mathrm{d}x_3 \wedge \mathrm{d}y_3 + \mathrm{d}a_3 \wedge \mathrm{d}b_3$.

Finally, on U_4 we set

$$x_4 + iy_4 = \rho_1 e^{i\zeta_1}, \quad a_4 + ib_4 = \rho_4 e^{i\zeta_4}, \quad \zeta_1 = \theta_1 - \theta_3 - k\theta_2, \quad \zeta_4 = \theta_4 - \theta_2,$$

and $\omega = \mathrm{d}x_4 \wedge \mathrm{d}y_4 + \mathrm{d}a_4 \wedge \mathrm{d}b_4$.

The normalized Hamiltonian function for ψ_t is $f = \pi\rho_1^2 - \kappa$, where κ is a constant determined by the condition $\int_M f\omega^2 = 0$. Straightforward calculations give

$$\int_M \omega^2 = \mu(2\tau - k\mu), \quad \text{and} \quad \int_M \pi\rho_1^2\omega^2 = \frac{\mu^2}{3}(3\tau - 2k\mu).$$

So

$$\kappa = \frac{\mu}{3}\left(\frac{3\lambda + k\mu}{2\lambda + k\mu}\right). \tag{4.4}$$

It is not easy to determine the transition function of $\det(TM)$ that corresponds to the coordinate transformation $(x_i, y_i, a_i, b_i) \to (x_j, y_j, a_j, b_j)$; that is why we will introduce polar coordinate on subsets of the domains U_j.

Given $0 < \epsilon << 1$, for $j = 1, 2, 3, 4$ we put

$$B_j = \{[z] \in U_j \, : \, |z_j| < 2\epsilon\} \quad \text{and} \quad B_0 = \{[z] \in M \, : \, |z_j| > \epsilon \text{ for all } j\}.$$

On B_0 the coordinates $(\frac{\rho_1^2}{2}, \varphi_1, \frac{\rho_2^2}{2}, \varphi_2)$ are well defined, and in these coordinates

$$\omega = \mathrm{d}\left(\frac{\rho_1^2}{2}\right) \wedge \mathrm{d}\varphi_1 + \mathrm{d}\left(\frac{\rho_2^2}{2}\right) \wedge \mathrm{d}\varphi_2.$$

On B_j $(j = 1, 2, 3, 4)$ we consider the Darboux coordinates (x_j, y_j, a_j, b_j) defined above. Then B_0, B_1, B_2, B_3, B_4 is a Darboux atlas for M. We assume that M is endowed with the orientation given by ω^2. This orientation agrees on B_0 with the one defined by $\mathrm{d}\rho_1^2 \wedge \mathrm{d}\varphi_1 \wedge \mathrm{d}\rho_2^2 \wedge \mathrm{d}\varphi_2$.

It is evident that $\psi_t(B_i) = B_i$, for $i = 0, 1, 2, 3, 4$. Since ψ_t on B_0 is simply the translation $\varphi_1 \to \varphi_1 + 2\pi t$ of the variable φ_1, the Maslov index J_0 of ψ in the trivialization defined on B_0 vanishes.

As B_j (for $j = 1, 2, 3, 4$) has "infinitesimal size" and $J_0 = 0$, the expression for I_ψ of Theorem 3 can be written as

$$I_\psi = \sum_{i<k} N_{ik} + O(\epsilon). \tag{4.5}$$

Since I_ψ is obviously independent of the coordinates, it follows from (4.5) that N_{ik} is independent, up to order ϵ, of the chosen Darboux coordinates in B_j, for $j = 1, 2, 3, 4$. Moreover N_{ik} with $0 \neq i < k$ is also of order ϵ.

On the other hand, if we replace B_j by

$$B'_j = \{[z] \in B_j \, : \, |z_r| > \epsilon, \; r \neq j\}$$

in the definition of N_{ik} (see Theorem 3) the new N_{ik} differs from the old one by a quantity of order ϵ. As on B'_1 the variable $\rho_2 \neq 0$, we can consider the Darboux coordinates

$$\left(x_1, y_1, \frac{\rho_2^2}{2}, \varphi_2\right)$$

on B'_1. Since $\rho_3 \neq 0$ on B'_2 we take the coordinates $(a_2, b_2, \frac{\rho_3^2}{2}, \xi_3)$ on B'_2. Similarly we will adopt the following coordinates: $(x_3, y_3, \frac{\rho_4^2}{2}, \chi_4)$ on B'_3 and $(\frac{\rho_1^2}{2}, \zeta_1, a_4, b_4)$ on B'_4.

Taking into account the preceding arguments

$$I_\psi = \sum_{j=1}^{4} N'_{0j} + O(\epsilon), \tag{4.6}$$

where

$$N'_{0j} = \frac{i}{\pi}\int_{A'_{0j}} f\, \mathrm{d}\log r_{0j} \wedge \omega \tag{4.7}$$

and

$$A'_{0j} = \{[z] \in M \,:\, |z_r| > \epsilon,\ \text{for all } r \neq j \text{ and } |z_j| = \epsilon\}.$$

The submanifold A'_{0j} is oriented as a subset of ∂B_0; that is, with the orientation induced by that of B_0.

Next we determine the value of N'_{01}. To know the transition function r_{01} one needs the Jacobian matrix R of the transformation

$$\left(x_1, y_1, \frac{\rho_2^2}{2}, \varphi_2\right) \to \left(\frac{\rho_1^2}{2}, \varphi_1, \frac{\rho_2^2}{2}, \varphi_2\right)$$

in the points of A'_{01}; with $\rho_1^2 = x_1^2 + y_1^2$, $\varphi_1 = \tan^{-1}(y_1/x_1)$. The non-trivial block of R is the diagonal one

$$\begin{pmatrix} x_1 & y_1 \\ r & s \end{pmatrix},$$

with $r = -y_1(x_1^2 + y_1^2)^{-1}$ and $s = x_1(x_1^2 + y_1^2)^{-1}$. The non-real eigenvalues of R are

$$\lambda_\pm = \frac{x_1 + s}{2} \pm \frac{i\sqrt{4 - (s + x_1)^2}}{2}.$$

On A'_{01} these non-real eigenvalues occur when $(s + x_1)^2 < 2$, that is, if $|\cos\varphi_1| < 2\epsilon(\epsilon^2 + 1)^{-1} =: \delta$. If $y_1 > 0$ then λ_- is of the first kind (see [10]) and λ_+ is of the first kind, if $y_1 < 0$.

Hence, on A'_{01},

$$\rho(R) = \begin{cases} \lambda_+|\lambda_+|^{-1} = x + iy, & \text{if } |\cos\varphi_1| < \delta \text{ and } y_1 < 0; \\ \lambda_-|\lambda_-|^{-1} = x - \mathrm{i}y, & \text{if } |\cos\varphi_1| < \delta \text{ and } y_1 > 0; \\ \pm 1, & \text{otherwise,} \end{cases}$$

where $x = \delta^{-1}\cos\varphi_1$, and $y = \sqrt{1 - x^2}$.

If we put $\rho(R) = e^{i\gamma}$, then $\cos\gamma = \delta^{-1}\cos\varphi_1$ (when $|\cos\varphi_1| < \delta$), and

$$\sin\gamma = \begin{cases} -\sqrt{1 - \cos^2\gamma}, & \text{if } \sin\varphi_1 > 0; \\ \sqrt{1 - \cos^2\gamma}, & \text{if } \sin\varphi_1 < 0. \end{cases}$$

So when φ_1 runs anticlockwise from 0 to 2π, γ goes round the circumference clockwise; that is, $\gamma = h(\varphi_1)$, where h is a function such that

$$h(0) = 2\pi, \quad \text{and} \quad h(2\pi) = 0. \tag{4.8}$$

As $r_{01} = \rho(R)$, we have $\mathrm{d}\log\, r_{01} = i dh$.

On A'_{10} the form ω reduces to $(1/2)\mathrm{d}\rho_2^2 \wedge \mathrm{d}\varphi_2$. From (4.7) one deduces

$$N'_{01} = \frac{i}{2\pi} \int_{A'_{01}} if \, \mathrm{d}h \wedge \mathrm{d}\rho_2^2 \wedge \mathrm{d}\varphi_2. \tag{4.9}$$

On the other hand according to the convention for orientations, $\{[z] : |z_1| = \epsilon\}$ as a subset of ∂B_0 is oriented by $-\mathrm{d}\varphi_1 \wedge \mathrm{d}\rho_2^2 \wedge \mathrm{d}\varphi_2$. And on A'_{01} the Hamiltonian function $f = -\kappa + O(\epsilon)$. Then it follows from (4.9) together with (4.8) that

$$N'_{01} = 2\tau\kappa + O(\epsilon). \tag{4.10}$$

The contributions $N'_{02}, N'_{03}, N'_{04}$ to I_ψ can be calculated in a similar way. One obtains the following results up to addends of order ϵ:

$$N'_{02} = 2\mu\kappa - \mu^2, \quad N'_{03} = 2\lambda(\kappa - \mu), \quad N'_{04} = \mu(2\kappa - \mu). \tag{4.11}$$

As I_ψ is independent of ϵ, it follows from (4.6), (4.10), (4.11) and (4.4) that

$$I_\psi = \frac{2k\mu^2}{3}\left(1 - \frac{\mu}{2\lambda + k\mu}\right).$$

Next we consider the loop $\tilde{\psi}$; the corresponding normalized Hamiltonian function is $\tilde{f} = \pi\rho_2^2 - \tilde{\kappa}$, where

$$\tilde{\kappa} = \frac{3\lambda^2 + 3k\lambda\mu + k^2\mu^2}{3(2\lambda + k\mu)}. \tag{4.12}$$

As in the preceding case

$$I_{\tilde{\psi}} = \sum_{j=1}^{4} \tilde{N}'_{0j} + O(\epsilon), \tag{4.13}$$

where

$$\tilde{N}'_{0j} = \frac{i}{\pi} \int_{A'_{0j}} \tilde{f} \, \mathrm{d}\log r_{0j} \wedge \omega.$$

The expression for $\tilde{N}'_{01}$ can be obtained from (4.9) by substituting f for $\tilde{f}$; so

$$\tilde{N}'_{01} = \tau(2\tilde{\kappa} - \tau) + O(\epsilon). \tag{4.14}$$

Analogous calculations give the following values for the $\tilde{N}'_{0j}$'s, up to summands of order ϵ

$$\tilde{N}'_{02} = 2\mu\tilde{\kappa}, \quad \tilde{N}'_{03} = \lambda(2\tilde{\kappa} - \lambda), \quad \tilde{N}'_{04} = \mu(2\tilde{\kappa} - k\mu - 2\lambda). \tag{4.15}$$

From (4.12)–(4.15) there follows the value for $I_{\tilde{\psi}}$ given in the statement of the theorem. □

Remark. In [2] it is proved that $\pi_1(\mathrm{Ham}(M)) = \mathbb{Z}$ when $k = 1$, therefore the quotient of I_ψ by $I_{\psi'}$, for arbitrary Hamiltonian loops of symplectomorphisms, is a rational number. For the particular loops considered in Theorem 8 the quotient $I_{\tilde{\psi}}/I_\psi$ equals $-k/2$, so Theorem 8 is consistent with the result of Abreu and McDuff.

Acknowledgement

This work was partially supported by Ministerio de Ciencia y Tecnología, grant MAT2003-09243-C02-00.

I thank Dusa McDuff for explaining to me properties of the Maslov index of the linearized flow, and Eva Miranda for clarifying me some points relating to action-angle variables.

References

[1] R. Abraham, J. Marsden, Foundations of Mechanics, Addison-Wesley, Reading, 1985.

[2] M. Abreu, D. McDuff, Topology of symplectomorphism groups of rational ruled surfaces, J. Amer. Math. Soc. 13 (2000) 971–1009.

[3] M. Audin, Torus Actions on Symplectic Manifolds, Birkhäuser, Basel, 2004.

[4] V. Guillemin, Moment Maps and Combinatorial Invariants of Hamiltonian T^n-spaces, Birkhäuser, Boston, 1994.

[5] V. Guillemin, L. Lerman, S. Sternberg, Symplectic Fibrations and Multiplicity Diagrams, Cambridge U.P., Cambridge, 1996.

[6] F. Lalonde, D. McDuff, L. Polterovich, Topological rigidity of Hamiltonian loops and quantum homology, Invent. Math. 135 (1999) 369–385.

[7] D. McDuff, D. Salamon, Introduction to Symplectic Topology, Clarendon Press, Oxford, 1998.

[8] L. Polterovich, Hamiltonian loops and Arnold's principle, Amer. Math. Soc. Transl. Ser. 2 180 (1997) 181–187.

[9] L. Polterovich, The Geometry of the Group of Symplectic Diffeomorphisms, Birkhäuser, Basel, 2001.

[10] D. Salamon, E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index, Comm. Pure Appl. Math. XLV (1992) 1303–1360.